LiTAM:
Formal and numerical equations

Stéphane Raynaud and Richard Kleeman

March 2003

1 Introduction

The goal of this model is to simulate the most important aspects of the climate in the tropical regions of Pacific and Atlantic, remaining in the range of complexity of an intermediate model.

Among important processes that have to be represented are :

- the initial basic structure of this model is based upon Gill [1980], a two-layer baroclinic and diagnostic model with surface heat forcing,
- in addition to the surface, we take into accound the diabatic source in the internal atmosphere following Kleeman [1991];
- the Ekman spiral within the boudary layer, responsible for the wind reorientation near the surface (certainly crucial for coupling in the atlantic region) is implementtedaccording to Neelin [1988].

Therefore, we solve the boundary layer by adding specific layers to an Gill type model, and we include an equation for the internal forcing at each layer.

Note: This documentation still does not describe the **moisture equation** responsible for the vertical profile of heat forcing (**Q**).

2 Formal equations

Momentum equations

On an equatorial beta plane

$$\begin{cases} -\beta y \mathbf{V} & = & -\mathbf{\Phi}_x + \mathbf{D_U}\mathbf{U} \\ \beta y \mathbf{U} & = & -\mathbf{\Phi}_y + \mathbf{D_U}\mathbf{V} \end{cases} , \tag{1}$$

where $\mathbf{U}$,$\mathbf{V}$ and $\mathbf{\Phi}$ are vertical vectors, functions of pressure p. $\mathbf{D_U}$ is the vertical diffusion operator

$$\mathbf{D_U} = \partial_p \nu \partial_p \mathbf{U} \ , \ \text{with } \nu\partial_p \mathbf{U} = -C\mathbf{U} \text{ as bottom boundary condition.} \tag{2}$$

ν is the vertical dissipation coefficient and depends on p.

Temperature equation

$$\omega\, \overline{\theta_p} = \mathbf{D}_\theta \theta^2 + \kappa \nabla \cdot \theta + (1000/p)^{2/7}\mathbf{Q} \ , \tag{3}$$

where κ is the horizontal coefficient of diffusion, $\mathbf{Q}$ is the diabatic heating rate (in $K.s^{-1}$), and $\mathbf{D}_\theta$ is the vertical diffusion following the formula:

$$\mathbf{D}_\theta \theta = \partial_p \nu \partial_p \theta \ , \ \text{with } \nu\partial_p \theta = -C\mathbf{U} \text{ as bottom boundary condition.} \tag{4}$$

Other vertical relationships

Using the ideal gas law, The geopotential is related to the temperature thanks to the equation:

$$\mathbf{\Phi}_p = -\frac{RT}{p} = \frac{R}{p}\left(\frac{p}{1000}\right)^{2/7} \theta \ , \tag{5}$$

where R is the constant of perfect gas.

The vertical velocity in pressure coordinates is expressed as:

$$\omega_p = -\nabla \cdot \mathbf{U} \ . \tag{6}$$

Without pertubation a the top of the atmosphere ($p = 0$), the integration from top to pressure p gives:

$$\omega = -\int_0^p \nabla \cdot \mathbf{U} \, dp \ . \tag{7}$$

3 Numerical equations

We will express all the equations versus Φ, main variable of the system.

3.1 The vertical grid

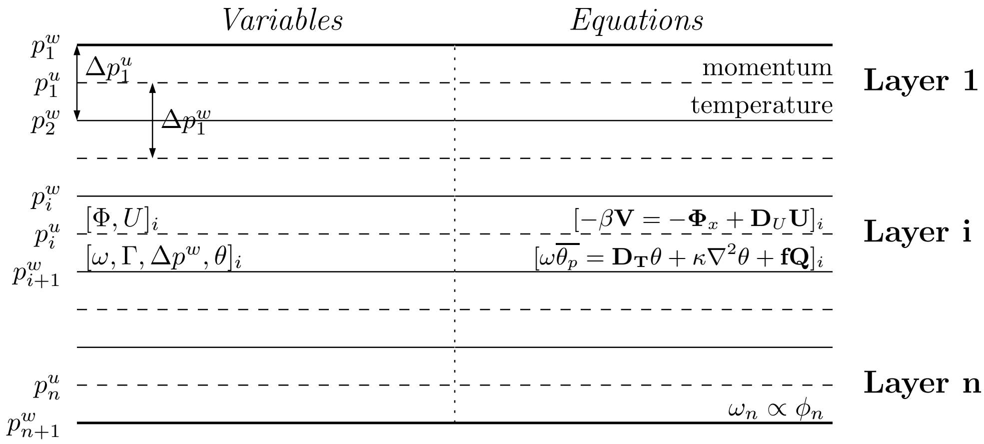

Figure 1: Vertical grid

3.2 Temperature equation

Geopotential and temperature

We have:

$$\mathbf{\Phi}_{i+1} - \mathbf{\Phi}_i = -\mathbf{\Gamma}_i \, \mathbf{\Theta}_i \ , \tag{8}$$

where:

$$\mathbf{\Gamma}_i = R \frac{\Delta p_i^w}{p_{i+1}^w} \left(\frac{p_{i+1}^w}{1000} \right)^{2/7} \ , \ i \in [1, n-1] \ .$$

Let's define $\mathbf{J}$ verifying:

$$\mathbf{\Theta} = \mathbf{J} \, \mathbf{\Phi} \ , \tag{9}$$

$$\mathbf{J} = \begin{pmatrix} \mathbf{\Gamma}_1^{-1} & -\mathbf{\Gamma}_1^{-1} & 0 & \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \mathbf{\Gamma}_n^{-1} & -\mathbf{\Gamma}_n^{-1} \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Vertical velocity

At the first level, and then integrating until level i, with a zero value at bottom:

$$\begin{array}{rcl} \omega_1 & = & \nabla \cdot \mathbf{U}_1 \\ \omega_i & = & \sum_{j=1}^{i} \nabla \cdot \mathbf{U}_j \\ \omega_n & = & 0 \end{array} \tag{10}$$

Vertical advection

Let's define $\mathbf{G}$ following:

$$\mathbf{G}\, \nabla \cdot \mathbf{U} = \omega\, \overline{\theta_p}\ , \tag{11}$$

$$\mathbf{G} = \begin{pmatrix} (\overline{\theta_p})_1 \Delta p_1^w & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & (\overline{\theta_p})_i \Delta p_i^w & \ddots & \vdots \\ \vdots & & \vdots & \ddots & 0 \\ (\overline{\theta_p})_n \Delta p_1^w & & (\overline{\theta_p})_n \Delta p_i^w & & (\overline{\theta_p})_n \Delta p_n^w \end{pmatrix}$$

Vertical diffusion

Following 4, the $\mathbf{D}_\theta$ matrix is:

$$\mathbf{D}_\theta = \begin{pmatrix} -\frac{\nu_2^u}{\Delta p_2^u \Delta p_1^w} & \frac{\nu_2^u}{\Delta p_2^u \Delta p_1^w} & 0 & \cdots & & \cdots & 0 \\ \frac{\nu_2^u}{\Delta p_2^u \Delta p_2^w} & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \ddots & \frac{\nu_i^u}{\Delta p_i^u \Delta p_i^w} & -\left(\frac{\nu_i^u}{\Delta p_i^u \Delta p_i^w} + \frac{\nu_{i+1}^u}{\Delta p_{i+1}^u \Delta p_i^w}\right) & \frac{\nu_{i+1}^u}{\Delta p_{i+1}^u \Delta p_i^w} & \ddots & 0 \\ \vdots & \ddots & & \ddots & \ddots & \ddots & \frac{\nu_n^u}{\Delta p_n^u \Delta p_{n-2}^w} \\ & & & & \ddots & -\frac{\nu_n^u}{\Delta p_n^u \Delta p_{n-1}^w} & \frac{\nu_n^u}{\Delta p_n^u \Delta p_{n-1}^w} \\ 0 & \cdots & & & \cdots & 0 & 1 \end{pmatrix} \tag{12}$$

3.3 Momentum equations

To facilitate the numerical solving, we find a basis of egeinvectors of the vertical dissipation operator $\mathbf{D_U}$. Vectors on this basis are noted with a ^{v}. ϵ is set of associated eigen values

$$\epsilon = \begin{pmatrix} \epsilon_1 & & 0 \\ & \ddots & \\ 0 & & \epsilon_n \end{pmatrix},$$

A is the matrix to change of base. Then, for example:

$$\mathbf{U^v} = \mathbf{A}\,\mathbf{U}\,, \quad \text{and} \quad \mathbf{D_U}\,\mathbf{U^v} = -\epsilon\,\mathbf{U^v}\,, \tag{13}$$

V and **Φ** obey to the same equation.

The momentum equation (1) becomes

$$\left\{ \begin{array}{rcl} -f\mathbf{V^v} & = & -\mathbf{\Phi^v}_x - \epsilon\mathbf{U^v} \\ f\mathbf{U^v} & = & -\mathbf{\Phi^v}_y - \epsilon\mathbf{V^v} \end{array} \right. , \tag{14}$$

with $f = \beta y$, y being the distance to the equator. Then

$$\left\{ \begin{array}{rcl} (f^2+\epsilon^2)\mathbf{U^v} & = & -\epsilon\mathbf{\Phi^v}_x - f\mathbf{\Phi^v}_y \\ (f^2+\epsilon^2)\mathbf{V^v} & = & f\mathbf{\Phi^v}_x - \epsilon\mathbf{\Phi^v}_y \end{array} \right. . \tag{15}$$

For the sake of simplicity, we choose to use substitute variables:

$$\begin{array}{rcl} \mathbf{B} & = & f^2+\epsilon^2 \\ \mathbf{X} & = & \epsilon\,\mathbf{B}^{-1} \\ \mathbf{Y} & = & 2\beta f\mathbf{B}^{-2} \\ \mathbf{Z} & = & \beta\mathbf{B}^{-1}(1-2f^2\mathbf{B}^{-1}) \end{array} .$$

We obtain

$$\left\{ \begin{array}{rcl} \mathbf{U^v}_x & = & \mathbf{B}^{-1}(-\epsilon\mathbf{\Phi^v}_{xx} - f\mathbf{\Phi^v}_{xx}) \\ \mathbf{V^v}_y & = & \beta\mathbf{B}^{-1}(1-2f^2\mathbf{B}^{-1})\,\mathbf{\Phi^v} + f\mathbf{B}^{-1}\mathbf{\Phi^v}_{xy} - \epsilon\mathbf{B}^{-1}\mathbf{\Phi^v}_{yy} - 2\epsilon\beta f\mathbf{B}^{-2}\mathbf{\Phi^v}_y \end{array} \right. ,$$

and thus, the divergence is expressed by

$$\nabla\cdot\mathbf{U^v} = -\mathbf{X}\,\Delta\mathbf{\Phi} + \mathbf{Z}\,\mathbf{\Phi}_x - \mathbf{Y}\,\mathbf{\Phi}_y \tag{16}$$

References

A. E. Gill. Some simple solutions for heat induced tropical circulation. *Quart. J. Roy. Meteor. Soc.*, 106:447–462, 1980.

R. Kleeman. A simple model of the atmospheric response to enso sea surface temperature anomalies. *J. Atm. Sc.*, 48:3–18, 1991.

J. D. Neelin. A simple model for surface stress and low-level flow in the tropical atmosphere driven by prescribed heating. *Quart. J. Roy. Meteor. Soc.*, 114:747–770, 1988.